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The non-linear Schrödinger model as a special continuum limit of the anisotropic Heisenberg model†

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Abstract. One-dimensional spin chains are investigated by constructing a mapping of the anisotropic Heisenberg model in the limit $\Delta \to 1$ onto the non-linear Schrödinger model (a gas of bosons with δ function interactions). Three applications of this mapping are given. First it is shown that the Bethe ansatz solutions of the anisotropic spin-$\frac{1}{2}$ Heisenberg model go over to the continuous Bethe ansatz solutions of the non-linear Schrödinger model. Then bound-state energies for arbitrary spin are calculated and as a third application a nearly identical mapping of the anisotropic spin-$\frac{1}{2}$ Heisenberg model in the limit $\Delta \to 0$ onto a non-linear Schrödinger model for fermion fields is given. Finally we construct an anisotropic SU($\mathcal{N}$) model. The continuum limit of this model we show to be the ($\mathcal{N}-1$)-component non-linear Schrödinger model, which can be solved by means of the nested Bethe ansatz.

1. Introduction

In this paper we investigate the mapping of the anisotropic Heisenberg model (XXZ model)

$$\mathscr{H}_{XXZ} = -\sum_{i=1}^{N} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta(S_i^z S_{i+1}^z - s^2)) + H_0 \sum_{i=1}^{N} S_i^z \tag{1}$$

onto the Hamiltonian of the so-called non-linear Schrödinger model (NLS model):

$$\mathscr{H}_{\mathrm{NLS}} = \int \mathrm{d}x(\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)). \tag{2}$$

In (1) S_i^x, S_i^y, S_i^z are the cartesian components of the spin operator at site i, Δ is the anisotropy parameter, H_0 parametrises the strength of the external magnetic field and $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is the spin quantum number.

In expression (2) $\Psi(x)$ is a non-relativistic boson field obeying canonical commutation relations

$$\begin{aligned} &[\Psi(x), \Psi^\dagger(y)] = \delta(x-y) \\ &[\Psi(x), \Psi(y)] = [\Psi^\dagger(x), \Psi^\dagger(y)] = 0 \end{aligned} \tag{3}$$

and $c \leqslant 0$ applies to the attractive two-body interaction.

Both models and their solutions by means of a Bethe ansatz are well known [1-5]. The Bethe ansatz solution of the XXZ model holds for $s = \frac{1}{2}$. For higher spin, the

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XXZ model as defined by (1) is not exactly integrable and is not well understood even in one dimension. In contrast, classical Heisenberg spin chains are again exactly soluble. The correspondence between classical anisotropic Heisenberg magnets and the non-linear Schrödinger equation has been discussed by Nakamura and Sasada [6] and Kundu and Pashaev [7]. In the quantum case the connection between XXZ model and NLS model has been noticed by Kulish and Sklyanin [8] in the extreme quantum limit $s=\frac{1}{2}$ within the context of the inverse scattering method and also by Schneider [9]. It is this correspondence which we will investigate in the following in detail.

For this purpose, first of all the spin operators on the linear chain have to be replaced by operators which obey the commutation relations of Bose operators. Several such transformations are known in the theory of spin waves—here we will use the Dyson-Maleev transformation. The continuum and weak-coupling limit of the anisotropic Heisenberg model yields, after a renormalisation of the energy, exactly the NLS model with the relation

$$\Delta = 1 - sca$$

between the coupling parameters Δ and c of the two models, and a is the lattice constant.

The outline of this paper is as follows. In § 2 the mapping of the XXZ model onto the NLS model is constructed. Eigenstates and eigenvalues of the NLS model are obtained by means of that mapping and are found to be in accordance with the direct solution of the NLS model by Thacker [1]. Next we investigate whether the mapping can be inverted. The calculations leading to the basic formulae (19) and (20) of the mapping are valid for arbitrary spin s. Accordingly the bound-state energies of the NLS model are used to derive an approximate formula for the bound-state energies of the weakly anisotropic ($\Delta - 1 \ll 1$) Heisenberg model for arbitrary spin s by means of the inverted mapping. In § 3 an analogous mapping of anisotropic SU($2s+1$) chains onto NLS models with $2s$ components is constructed. These SU($2s+1$) chains represent a different approach to higher spin as compared to (1). They may be rewritten in terms of spin-s SU(2) chains with Schrödinger-type exchange interaction instead of the usual Heisenberg-Dirac-Van Vleck exchange interaction. The results are discussed in § 4.

2. Mapping of the *XXZ* model onto the NLS model

2.1. *Occupation number formalism for the Heisenberg spin chain*

The states of a one-dimensional spin chain with respect to the action of the anisotropic Heisenberg (or XXZ) Hamiltonian (1) for $s=\frac{1}{2}, 1, \frac{3}{2}, \ldots,$ can be classified by means of an occupation number formalism. Taking as a reference (or vacuum) state the state with all spins aligned, the number n_i with values between 0 and $2s$

$$n_i = m_i + s \qquad m_i = \text{eigenvalue of } S_i^z \tag{4}$$

counts the spin deviations at site i of the chain and is a suitable occupation number (cf figure 1). In addition, the Hamiltonian (1) commutes with the operator S^z_{op}, the z component of the total spin, $S^z_{\text{op}} = \Sigma_i S^z_i$. Therefore the common eigenstates can be classified according to the eigenvalues of the operator S^z_{op}. Eigenstates with specified

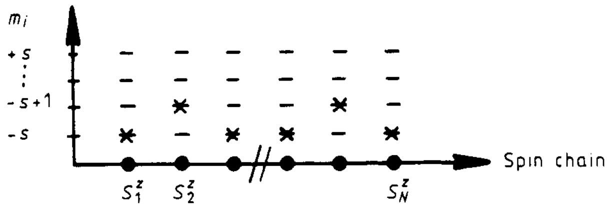

Figure 1. A state of the spin chain containing two spin deviations. A certain superposition of such states leads to a 2-magnon eigenstate.

n, where

$$n = S^z + Ns \qquad S^z = \text{eigenvalue of } S^z_{\text{op}} \tag{5}$$

are called n-magnon states.

This classification scheme for the states emerges naturally, if, instead of the spin operators S_i^x, S_i^y, S_i^z with their commutation relations

$$[S^i_n, S^j_m] = \mathrm{i}\, \varepsilon_{ijk} S^k_n \delta_{nm} \qquad n, m \in N \tag{6}$$

ascending and descending operators are defined

$$S_i^+ = S_i^x + \mathrm{i}\, S_i^y \qquad S_i^- = S_i^x - \mathrm{i}\, S_i^y \tag{7}$$

obeying the commutation relations

$$[S_i^z, S_i^+] = S_i^+ \qquad [S_i^z, S_i^-] = -S_i^- \qquad [S_i^+, S_i^-] = 2S_i^z. \tag{8}$$

The action of these operators on the spin states is given by

$$\begin{aligned} S_i^+|n_i\rangle &= [(2s-n_i)(n_i+1)]^{1/2}|n_i+1\rangle \\ S_i^-|n_i+1\rangle &= [(2s-n_i)(n_i+1)]^{1/2}|n_i\rangle \\ S_i^z|n_i\rangle &= (n_i - s)|n_i\rangle. \end{aligned} \tag{9}$$

That implies S_i^z characterises the state at site i, S_i^+ creates spin deviations and S_i^- annihilates spin deviations. In terms of these operators, the Hamiltonian (1) can be written as

$$\mathscr{H}_{XXZ} = -\sum_{i=1}^{N} \left(\tfrac{1}{2}(S_{i+1}^- S_i^+ + S_{i+1}^+ S_i^-) + \Delta(S_i^z S_{i+1}^z - s^2)\right) + H_0 \sum_{i=1}^{N} S_i^z. \tag{10}$$

It is impossible to diagonalise $\mathscr{H}$ by a canonical transformation, but it is possible to transform to new sets of variables, being either pure Bose or pure Fermi operators, so that $\mathscr{H}$ maintains a relatively simple form. In the ensuing section the Dyson–Maleev transformation to Bose operators will be applied to this Hamiltonian. The Hamiltonian then describes a one-dimensional chain with boson-like excitations. In a special continuum limit this system can be mapped onto the Hamiltonian of the NLS model.

2.2. Dyson–Maleev transformation to Bose operators

A detailed treatment of this transformation may be found in the textbook by Akhiezer *et al* [10]. The most important point is: a one-dimensional chain with boson-like excitations admits more states than a corresponding spin chain, because infinitely many bosons may be created above one site, whereas only $2s$ spin deviations are possible in the original model (cf figure 2).

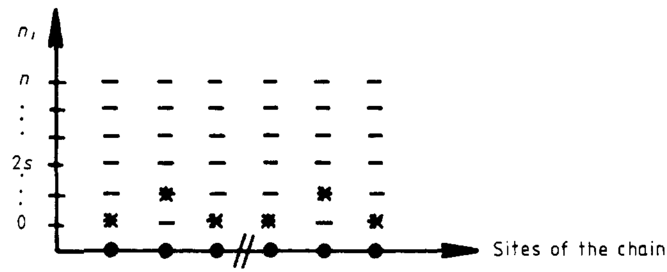

Figure 2. A one-dimensional chain with boson-like excitations.

These additional states of the Bose chain, however, do not affect our procedure, because a given state of the spin chain is automatically mapped onto a corresponding state of the Bose chain located in the sector with excitations between 0 and $2s$ per site, and in the following we are only interested in that part of the Hilbert space.

The Dyson–Maleev transformation leading to Bose operators is given by

$$\begin{aligned}\tilde{S}_i^+ &= (2s)^{1/2} a_i^+ \left(1 - \frac{a_i^+ a_i}{2s}\right)\\ \tilde{S}_i^- &= (2s)^{1/2} a_i \\ \tilde{S}_i^z &= -s + a_i^+ a_i.\end{aligned} \tag{11}$$

It is easy to check that if the a_i, a_i^+ obey canonical commutation relations for Bose operators

$$\begin{aligned}&[a_i, a_j^+] = \delta_{i,j} \qquad \delta_{i,j} \text{ is the Kronecker symbol}\\ &[a_i^+, a_j^+] = [a_i, a_j] = 0\end{aligned} \tag{12}$$

then the $\tilde{S}_i^+$, $\tilde{S}_i^-$ and $\tilde{S}_i^z$ on the left-hand side fulfil the same commutation relations as the spin operators S_i^+, S_i^- and S_i^z. Note from (11) that a_i, a_i^+ and $\tilde{S}_i^-$, $\tilde{S}_i^+$ cannot at the same time be Hermitian conjugates. This inherent difficulty of the Dyson–Maleev transformation plays no role here, for the Hamiltonian (10) contains only terms bilinear in S_i^+ and S_i^- and hence in a_i^+ and a_i (the distinction between i and $i+1$ will vanish in the continuum limit). Accordingly the spin operators in (10) may be replaced by the tilded spin operators.

Applying the Dyson–Maleev transformation (11) to $\mathscr{H}$, one obtains the representation

$$\begin{aligned}\mathscr{H}_{XXZ} = {}& -H_0 N s + (2s\Delta + H_0) \sum_{i=1}^{N} a_i^+ a_i - s \sum_{i=1}^{N} (a_{i+1} a_i^+ + a_{i+1}^+ a_i)\\ &+ \sum_{i=1}^{N} [\tfrac{1}{2}(a_{i+1} a_i^+ a_i^+ a_i + a_{i+1}^+ a_{i+1}^+ a_{i+1} a_i) - \Delta a_{i+1}^+ a_{i+1} a_i^+ a_i].\end{aligned} \tag{13}$$

2.3. *The continuum limit*

The intended limit will cause a transmutation of the discrete chain so far considered with N spins to a continuum model of length $L = Na$, where a is the lattice constant.

This is achieved by subjecting the Bose operators defined at discrete lattice sites to a Fourier transformation

$$a_x = \frac{1}{\sqrt{N}} \sum_k \exp(-\mathrm{i}kx) a_k \tag{14}$$

and carrying out the usual limiting procedure within the Fourier representation

$$\sum_k \to \frac{L}{2\pi} \int \mathrm{d}k \qquad L = Na \qquad a_k \to \left(\frac{2\pi}{L}\right)^{1/2} \Psi(k). \tag{15}$$

Finally one transforms back to real space by means of the inverse Fourier transformation

$$\Psi(k) = \frac{1}{(2\pi)^{1/2}} \int \mathrm{d}x \exp(\mathrm{i}kx)\Psi(x). \tag{16}$$

This three-step procedure results in the new Hamiltonian

$$\begin{aligned} \mathscr{H}_{XXZ} = & -H_0 Ns + (2s\Delta + H_0) \int \mathrm{d}x\, \Psi^\dagger(x)\Psi(x) \\ & - s \int \mathrm{d}x(\Psi^\dagger(x+a)\Psi(x) + \Psi^\dagger(x-a)\Psi(x)) \\ & + a \int \mathrm{d}x[\tfrac{1}{2}(\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x+a) + \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x-a)) \\ & - \Delta\Psi^\dagger(x+a)\Psi^\dagger(x)\Psi(x+a)\Psi(x)]. \end{aligned} \tag{17}$$

Here $\Psi(x)$, $\Psi^\dagger(x)$ are Bose field operators fulfilling the canonical commutation relations (3). Making use of $\Delta = 1 + (\Delta - 1)$ and Taylor expanding $\Psi(x \pm a)$ and $\Psi^\dagger(x \pm a)$ in powers of the lattice constant a, the following representation for $\mathscr{H}$ is easily obtained

$$\begin{aligned} \mathscr{H}_{XXZ} = & H_0 S^z_{\mathrm{transf}} + 2s(\Delta - 1) \int \mathrm{d}x\, \Psi^\dagger(x)\Psi(x) + sa^2 \int \mathrm{d}x\, \partial_x\Psi^\dagger(x)\partial_x\Psi(x) \\ & - a(\Delta - 1) \int \mathrm{d}x\, \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) + \mathrm{O}(a^3) \end{aligned} \tag{18}$$

where $H_0 S^z_{\mathrm{transf}} \equiv H_0 \int \mathrm{d}x\, \Psi^\dagger(x)\Psi(x) - H_0 Ns$ has been introduced as an abbreviation for the terms resulting from $H_0 S^z_{\mathrm{op}}$ after the various transformations.

The individual terms in (18) exhibit remarkable similarities with the non-linear Schrödinger model defined by (2), but in the ordinary continuum limit $a \to 0$, $N \to \infty$, $L = Na =$ constant, $\mathscr{H} \to \mathscr{H}/sa^2$, the interaction term would diverge.

The desired representation of the NLS Hamiltonian as a special continuum limit of the *XXZ* Hamiltonian is obtained by means of the following scaling of the anisotropy parameter Δ:

$$\Delta = 1 - sca + \tfrac{1}{2}m^2a^2. \tag{19}$$

Only then the continuum limit for $\mathscr{H}$ given by (18) with simultaneous subtraction of divergent terms results in

$$\begin{aligned} & \frac{\mathscr{H}_{XXZ} - H_0 S^z_{\mathrm{transf}} + 2s^2 ca \int \mathrm{d}x\, \Psi^\dagger(x)\Psi(x)}{sa^2} \\ & \xrightarrow[a \to 0]{} \int \mathrm{d}x(\partial_x\Psi^\dagger(x)\partial_x\Psi(x) + m^2\Psi^\dagger(x)\Psi(x) + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)) \end{aligned} \tag{20}$$

and hence, for $m^2 = 0$, yields exactly the non-linear Schrödinger model of equation (2). For $m^2 \neq 0$ the model contains in addition a rest mass term.

The formulae (19) and (20) establish the NLS model as a special continuum limit of the anisotropic Heisenberg model for arbitrary spin s. Note that the Zeeman term appears in (20) only in the subtracted terms and effects no change at all. The external magnetic field H_0 might also be scaled, but resulting in less interesting Hamiltonians. Finally we point out that the mapping of the anisotropic Heisenberg model's Hilbert space into the considerably larger Hilbert space of the NLS model is injective and the correspondence established above applies only to the low-energy excitations of the NLS model.

2.4. *Energy eigenvalues and Bethe ansatz eigenstates of the NLS model*

We now present a first application of the mapping constructed in §§ 2.1-2.3. In order to obtain the Bethe ansatz solution of the NLS model (given previously by Thacker [1]) the mapping procedure can be applied systematically to the Bethe ansatz solution of the spin-$\frac{1}{2}$ XXZ model (given by Yang and Yang [3], des Cloizeaux and Gaudin [4] and Gochev [5]). We understand this as an example of how to obtain the Bethe ansatz solution of a quantum field theoretical model in a simple and systematical way from a corresponding discrete model. The corresponding elements of the Bethe ansatz solutions of the two models are listed in table 1.

2.5. *Bound states of the Heisenberg ferromagnet for arbitrary spin*

We now present a second application of the mapping constructed in §§ 2.1-2.3. In the foregoing sections the mapping of the anisotropic Heisenberg model onto the non-linear Schrödinger model has been discussed. As a matter of fact, it has been shown that the Bethe ansatz for the spin-$\frac{1}{2}$ XXZ model is transformable to a Bethe ansatz for the NLS model and that the well known eigenstates and eigenvalues of the NLS model can thus be reproduced. The question arises whether it is possible to invert this mapping and to make use of the well known eigenstates and eigenvalues of the NLS model in order to learn more about the XXZ model for general s. That is indeed possible.

Following the argument of Schneider [9], the continuum limit can be used to study features of the discrete model at the point $P=0$, i.e. in the centre of the first Brillouin zone. In particular the n-boson bound-state energy can be used to derive a formula for the n-magnon bound state energies of the Heisenberg ferromagnet for general s, valid for small anisotropies $\Delta - 1 \ll 1$, and $P=0$.

In order to invert the mapping, the continuum limit (19) and (20) for the energy eigenvalues ($m=0$)

$$\begin{aligned} &\Delta = 1 - sca \\ &\frac{E_{XXZ} - \text{constant}}{sa^2} \xrightarrow[a\to 0]{} E_{\text{NLS}} \qquad (21) \\ &\text{constant} \equiv H_0 S^z - 2s^2 nca \end{aligned}$$

is approximated by an equation that can be solved for E_{NLS}, i.e.

$$\begin{aligned} &\Delta - 1 = -sca \\ &\frac{E_{XXZ} - \text{constant}}{sa^2} \underset{\Delta-1\ll 1}{\simeq} E_{\text{NLS}}. \end{aligned} \qquad (22)$$

Table 1. Corresponding elements of the Bethe ansatz solutions of the spin-$\frac{1}{2}$ XXZ and NLS models under the mapping given by formulae (19) and (20).

spin-$\frac{1}{2}$ XXZ model	NLS model
Hamiltonian	Hamiltonian
$\mathscr{H}_{XXZ} = -\sum_{i=1}^{N} [S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta(S_i^z S_{i+1}^z - \frac{1}{4})] + H_0 \sum_{i=1}^{N} S_i^z$	$\mathscr{H}_{\rm NLS} = \int dx(\partial_x \Psi^\dagger(x)\partial_x \Psi(x) + m^2\Psi^\dagger(x)\Psi(x) + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x))$
n-magnon energies	n-boson energies
$E_{XXZ} = \sum_{j=1}^{n} (\Delta - \cos k_j) + H_0 S^z$	$E_{\rm NLS} = \sum_{j=1}^{n} k_j^2 + nm^2$
$\exp(ik_jN) = (-1)^{n-1}\exp\left(-i\sum_l \Theta(k_j, k_l)\right)$	$\exp(ik_jL) = \exp\left(-i\sum_l \Delta(k_j - k_l)\right)$
$j = 1, 2, \ldots, n$	$j = 1, 2, \ldots, n$
$\Theta(p, q)$: scattering phase shift	$\Delta(p-q)$: scattering phase shift
$\exp(i\Theta(p,q)) = \dfrac{2\Delta\, e^{iq} - 1 - \exp[i(p+q)]}{2\Delta\, e^{ip} - 1 - \exp[i(p+q)]}$	$\exp[i\Delta(p-q)] = \dfrac{p-q-ic}{p-q+ic}$
n-magnon eigenstates	n-boson eigenstates
$\lvert k_1 k_2 \ldots k_n\rangle_{BA} = \sum_{m<m_2,\ldots<m_n} A_{m_1 m_2 \ldots m_n} \lvert S_{m_1}^+ S_{m_2}^+ \ldots S_{m_n}^+\lvert 0\rangle$	$\lvert k_1 k_2 \ldots k_n\rangle_{BA} = \int \left(\prod_{i=1}^{n} \exp(ik_i x_i)\, dx_i\right) \prod_{i<j\leq n}\left(1 - \frac{ic}{k_i - k_j}\varepsilon(x_i - x_j)\right)\Psi_{x_1}^\dagger \Psi_{x_2}^\dagger \ldots \Psi_{x_n}^\dagger \lvert 0\rangle$
$A_{m_1 m_2 \ldots m_n} \equiv \sum_P A_P \exp\left(\sum_{j=1}^{n} k_{P(j)} m_j\right)$	
$P(1), P(2), \ldots, P(n)$ is a permutation of $1, 2, \ldots, n$ and $k_1, k_2, \ldots, k_n$ are n distinct real or complex numbers.	$\varepsilon(x_i - x_j) \equiv \Theta(x_i - x_j) - \Theta(x_j - x_i)$ is the double-step function and $k_1, k_2, \ldots, k_n$ are n distinct real or complex numbers.
n-magnon bound-state energies for $\Delta > 1$	n-boson bound-state energies for $c \leq 0$
$E_{XXZ,n}(P) = H_0 S^z + \dfrac{\sinh\phi}{\sinh n\phi}(\cosh n\phi - \cos aP)$	$E_{{\rm NLS},n}(P) = \dfrac{P^2}{n} - \dfrac{c^2}{12} n(n^2-1) + nm^2$
$\cosh\phi \equiv \Delta \qquad 0 \leq P \leq 2\pi \qquad n = 1, 2, \ldots, N.$	$0 \leq P \leq 2\pi, \quad n = 1, 2, \ldots.$

Obviously the approximation (22) is only valid in the limit of weak anisotropy $\Delta - 1 \ll 1$, but for arbitrary s and arbitrary c.

Using formula (22) and the case $m = 0$ from table 1 one obtains for the energies of n-magnon bound states of the Heisenberg ferromagnet in the continuum and weak anisotropy limits for $P = 0$:

$$\begin{aligned} E_{XXZ,n,P=0} &= sa^2 E_{{\rm NLS},n,P=0} + H_0 S^z - 2s^2 nca \\ &= sa^2\left(-\frac{c^2}{12} n(n^2-1)\right) + H_0 S^z - 2s^2 nca \\ &= H_0 S^z + 2s\left(n(\Delta-1) - \frac{(\Delta-1)^2 n(n^2-1)}{24s^2}\right). \end{aligned} \tag{23}$$

This expression, being valid for arbitrary s, can be compared with the known exact bound-state energies for $s = \frac{1}{2}$ listed in table 1:

$$E_{n,P=0,s=\frac{1}{2}} = H_0 S^z + \frac{\sinh \phi}{\sinh n\phi}(\cosh n\phi - 1)$$

$$= H_0 S^z + n(\Delta - 1) - \frac{(\Delta-1)^2 n(n^2-1)}{6} + \mathrm{O}(\Delta-1)^3. \tag{24}$$

Comparison shows that expression (23) agrees for $s=\frac{1}{2}$ to second order in $(\Delta-1)$ with the Taylor expanded exact result (24). That means the continuum limit permits us to calculate the exact bound-state expressions at $P=0$ in the appropriate weak anisotropy limit $(\Delta - 1 \ll 1)$.

2.6. *Mapping of the spin-$\frac{1}{2}$ XXZ model onto a fermion field theory with δ function interaction*

As a third application of the mapping procedure developed in §§ 2.1-2.3 we note that the well known transformation of the spin-$\frac{1}{2}$ *XXZ* model onto a chain of interacting spinless fermions by means of a Jordan-Wigner transformation [11-15] can be used to derive the Bethe ansatz solution of a continuous Fermi field theory. The continuum limit is reached by means of the procedure explained in § 2.3. This results in the Hamiltonian

$$\mathcal{H}_{XXZ} = -\tfrac{1}{2}\Delta N + (\Delta - 1)\int \mathrm{d}x\, \Phi^\dagger(x)\Phi(x) + \tfrac{1}{2}a^2 \int \mathrm{d}x\, \partial_x\Phi^\dagger(x)\partial_x\Phi(x)$$

$$- a\Delta \int \mathrm{d}x\, \Phi^\dagger(x)\Phi^\dagger(x)\Phi(x)\Phi(x) + \mathrm{O}(a^3) \tag{25}$$

where the $\Phi(x)$, $\Phi^\dagger(x)$ are Fermi field operators obeying canonical anticommutation relations. The various terms in (25) are multiplied with different powers of a. Only by introducing the parametrisation

$$\Delta = -\tfrac{1}{2}ca + \tfrac{1}{2}m^2a^2 \tag{26}$$

can a fermion field theory with non-trivial interactions result in the limit $N\to\infty$, $a\to 0$, $L = Na = \text{constant}$. In that case we obtain $(E_0 = -\frac{1}{2}\Delta N)$:

$$\frac{\mathcal{H}_{XXZ} - E_0 + (1+\frac{1}{2}ca)\int \mathrm{d}x\, \Phi^\dagger(x)\Phi(x)}{\frac{1}{2}a^2}$$

$$\xrightarrow[a\to 0]{} \int \mathrm{d}x(\partial_x\Phi^\dagger(x)\partial_x\Phi(x) + m^2\Phi^\dagger(x)\Phi(x) + c\Phi^\dagger(x)\Phi^\dagger(x)\Phi(x)\Phi(x)). \tag{27}$$

Formulae (26) and (27) establish the fermion field theory defined by

$$\mathcal{H}_\mathrm{F} = \int \mathrm{d}x(\partial_x\Phi^\dagger(x)\partial_x\Phi(x) + m^2\Phi^\dagger(x)\Phi(x) + c\Phi^\dagger(x)\Phi^\dagger(x)\Phi(x)\Phi(x)) \tag{28}$$

as a special continuum limit of the *XXZ* model in the neighbourhood of the point $\Delta = 0$. The complete Bethe ansatz solution of the spin-$\frac{1}{2}$ *XXZ* model is mapped onto a Bethe ansatz solution for $\mathcal{H}_\mathrm{F}$.

3. The multicomponent NLS model as a continuum limit of anisotropic SU($\mathcal{N}$) models

We now proceed to construct analogues of the *XXZ* Hamiltonian with SU($\mathcal{N}$) symmetry. The continuum limit of these is shown to be the multicomponent non-linear Schrödinger model solved by Yang [16]. That is achieved by extending our previous treatment of this problem from SU(2) to SU($\mathcal{N}$). We study the case SU(3) in detail: the closest conceivable analogue of Hamiltonian (1) for the SU(3) case is the operator

$$H_1 = -2 \sum_{i=1}^{N} \sum_{\alpha=1}^{8} \lambda_\alpha T_{\alpha,i} T_{\alpha,i+1} + \tfrac{2}{3} \lambda N \tag{29}$$

with parameters $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5 = 1$, $\lambda_3 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda$. Here the T_α are the generators of the SU(3) Lie algebra satisfying the commutation relations $[T_\alpha, T_\beta] = \mathrm{i} \sum_{\gamma=1}^{8} f_{\alpha\beta\gamma} T_\gamma$ with completely antisymmetric structure constants $f_{\alpha\beta\gamma}$ (tabulated, e.g., in ch VII of reference [17]). Equation (29) can be rewritten in the form

$$H_1 = -2 \sum_{i=1}^{N} \Bigg(\sum_{\alpha=\alpha_1,\alpha_3} (E^+_{\alpha,i} E^-_{\alpha,i+1} + E^-_{\alpha,i} E^+_{\alpha,i+1}) + \lambda (E^+_{\alpha_2,i} E^-_{\alpha_2,i+1} + E^-_{\alpha_2,i} E^+_{\alpha_2,i+1}) \\ + \lambda (T_{3,i} T_{3,i+1} + T_{8,i} T_{8,i+1} - \tfrac{1}{3}) \Bigg). \tag{30}$$

Here T_3 and T_8 represent the two generators forming the Cartan subalgebra of SU(3), $E^+_{\boldsymbol{\alpha}}$ and $E^-_{\boldsymbol{\alpha}}$ are raising and lowering operators for the weights and the $\boldsymbol{\alpha}$ represent root vectors $\boldsymbol{\alpha}^1 = (1/2, \sqrt{3}/2)$, $\boldsymbol{\alpha}^2 = (1/2, -\sqrt{3}/2)$, $\boldsymbol{\alpha}^3 = (1, 0)$. The transformation of operators is given by

$$E^\pm_{\boldsymbol{\alpha}^1} = \frac{1}{\sqrt{2}} (T_4 \pm \mathrm{i} T_5) \qquad E^\pm_{\boldsymbol{\alpha}^2} = \frac{1}{\sqrt{2}} (T_6 \pm \mathrm{i} T_7) \qquad E^\pm_{\boldsymbol{\alpha}^3} = \frac{1}{\sqrt{2}} (T_1 \pm \mathrm{i} T_2). \tag{31}$$

For details see [17].

Using the defining representation of SU(3) generated by the 3×3 matrices T_α explained above, the three basis states are obtained from the state with highest weight vector $|1/2, 1/(2\sqrt{3})\rangle$ by application of $E^-_{\boldsymbol{\alpha}^1}$ and $E^-_{\boldsymbol{\alpha}^3}$. In figure 3 the weight diagram is plotted together with the action of the raising and lowering operators onto the states

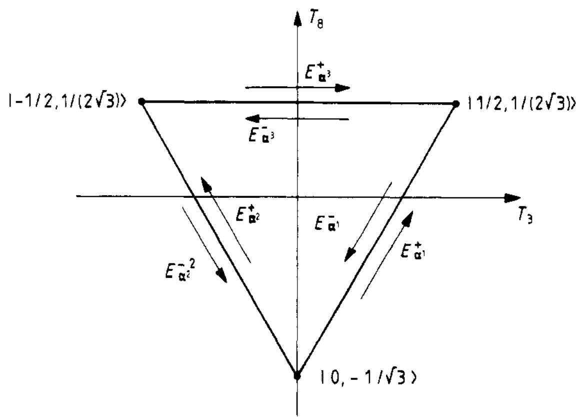

Figure 3. SU(3) weight diagram. States are represented by their weight vectors and it is shown how the states can be transformed into each other by application of the raising and lowering operators $E^\pm_{\boldsymbol{\alpha}^i}$.

represented by their weight vectors $\boldsymbol{\beta}^i$, $i=1,2,3$. Operators not explicitly indicated between two neighbouring states annihilate these states.

The generators of the SU(3) Lie algebra may be represented in the usual way by either three coupled bosons or three coupled fermions according to

$$\chi_\alpha \equiv \sum_{i,j} b_i^+[T_\alpha]_{ij} b_j \tag{32}$$

where $[\chi_\alpha, \chi_\beta] = i\sum_c f_{abc}\chi_c$, provided that

$$[b_i, b_j^+]_\pm = \delta_{ij} \qquad [b_i^+, b_j^+]_\pm = [b_i, b_j]_\pm = 0. \tag{33}$$

Using the fact that $b_1^+b_1 + b_2^+b_2 + b_3^+b_3 = 1$, one may eliminate one of the bosons (fermions). This defines a SU(3) analogue of the Holstein-Primakoff or Dyson-Maleev transformation. We use the latter transformation to represent the set of $E_{\boldsymbol{\alpha}}^\pm$ operators by two bosons (fermions) b_1, b_2 according to

$$\begin{aligned}
\sqrt{2}E^+_{\boldsymbol{\alpha}_1} &= (1 - b_1^+b_1 - b_2^+b_2)b_1\\
\sqrt{2}E^-_{\boldsymbol{\alpha}_1} &= b_1^+\\
\sqrt{2}E^+_{\boldsymbol{\alpha}_3} &= (1 - b_1^+b_1 - b_2^+b_2)b_2\\
\sqrt{2}E^-_{\boldsymbol{\alpha}_3} &= b_2^+\\
\sqrt{2}E^+_{\boldsymbol{\alpha}_2} &= b_2^+b_1\\
\sqrt{2}E^-_{\boldsymbol{\alpha}_2} &= b_1^+b_2\\
2T_3 &= 1 - b_1^+b_1 - 2b_2^+b_2\\
2T_8 &= (1/\sqrt{3})(1 - 3b_1^+b_1).
\end{aligned} \tag{34}$$

Inserting (34) into (30), a representation of the Hamiltonian H_1 in terms of a chain of bosons (fermions) is obtained which is manifestly symmetric with respect to the two bosons (fermions) b_1 and b_2. The continuum limit can be studied by applying the procedure of § 2.3, i.e. formulae (14)-(20), step by step. This yields

$$\frac{H_1 - \int \mathrm{d}x\, \Sigma_{i=1,2}\, 2(\lambda-1)\psi_i^\dagger\psi_i}{a^2} \underset{a\to 0}{\longrightarrow} \int \mathrm{d}x \left(\sum_{i=1,2} \partial_x\psi_i^\dagger(x)\partial_x\psi_i(x) + c \sum_{i,j=1,2} \psi_i^\dagger(x)\psi_j^\dagger(x)\psi_j(x)\psi_i(x) \right) \tag{35}$$

provided that the scaling relation

$$\lambda = 1 - \tfrac{1}{2}ca \tag{36}$$

is used, where a is the lattice constant. Thus the two-component NLS model with either boson or fermion fields $\psi_i(x)$ is shown to be a special continuum limit of the SU(3) chain with anisotropic interactions. By using $\lambda = 1 - \frac{1}{2}ca + \frac{1}{2}m^2a^2$ instead of (36), a rest mass term will survive the continuum limit in the same way as in (20).

It is obvious how to generalise the above calculations to SU($\mathcal{N}$). The state with highest weight (any weight can be used as a highest weight) is chosen as a formal vacuum state and the $\mathcal{N}-1$ lowering operators $E^-_{\boldsymbol{\alpha}}$ acting on it become $\mathcal{N}-1$ elementary bosons (fermions) b_i, corresponding to $\mathcal{N}-1$ boson fields (fermion fields) $\psi_i(x)$ in the continuum limit. The anisotropy is induced by multiplying the remaining $(\mathcal{N}-1)\times(\mathcal{N}-2)/2$ exchange terms $E^+_{\boldsymbol{\beta},i}E^-_{\boldsymbol{\beta},i+1} + E^-_{\boldsymbol{\beta},i}E^+_{\boldsymbol{\beta},i+1}$ by $\lambda \neq 1$. These $(\mathcal{N}-1)(\mathcal{N}-2)/2$

operators are associated with the vertices of the $(\mathcal{N}-2)$ simplex obtained by omitting the state with highest weight. The terms associated with operators H_γ of the Cartan subalgebra are likewise to be multiplied by λ.

The anisotropic SU$(2s+1)$ chain operator is of a form similar to (30), i.e.

$$H_s=-2\sum_{i=1}^{N}\Bigg(\sum_{\alpha}(E^+_{\alpha,i}E^-_{\alpha,i+1}+E^-_{\alpha,i}E^+_{\alpha,i+1})$$
$$+\lambda\sum_{\beta}(E^+_{\beta,i}E^-_{\beta,i+1}+E^-_{\beta,i}E^+_{\beta,i+1})+\lambda\sum_{\gamma}H_{\gamma,i}H_{\gamma,i+1}-\lambda/3\Bigg) \tag{37}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ sum over the roots indicated above.

In our special continuum limit this model yields the $(\mathcal{N}-1)$-component NLS model solved by Yang [16] by means of a nested Bethe ansatz. The second quantised operator is obtained in the form (35) with the replacement $i, j=1, 2, \ldots, \mathcal{N}-1$.

4. Summary and discussion

In this paper various mappings have been constructed and applied relating the XXZ and NLS Hamiltonians and their respective solutions. There is hardly a problem with one direction of the mapping (XXZ model $\rightarrow$ NLS model), the only difficulty being the Dyson-Maleev transformation with the additional states of the boson Hilbert space. However, the mapping between the physical states is exact and the NLS model with its proper commutation relations emerges and Thacker's Bethe ansatz solution of the NLS model [1] is reproduced.

In the reverse direction of the mapping (NLS model $\rightarrow XXZ$ model) there are in addition to the notorious problem with the unphysical states two further problems: there are two limiting procedures which can only approximately be reverted. First the continuum limit (15) which implies that all reverted results hold only for long wavelengths ($k\approx 0$) and then the special rescaling limit (19) and (20) which is unavoidable in order to force the $O(a^3)$ terms to approach zero. These terms cannot be reconstructed in the inversion, rendering the bound-state energies (23) undetermined in higher orders. In the extreme quantum case $s=\frac{1}{2}$ the results of Kulish and Sklyanin [8] are recovered by specialising formulae (19) and (20) of the present work to that case. To summarise: it is found that it is possible to 'extract' the NLS model from the more complicated XXZ model by means of the formal expansion (18) and the special continuum and rescaling limits (19) and (20). Most of the information concerning the non-linear dynamics of the Heisenberg model is lost in these limits, whereas the earlier formulae (13) or (18) still contain the full dynamics within the $O(a^3)$ terms—which are not required to be small! The expansion in powers of a is only a formal one and is solely introduced in order to isolate the NLS dynamics from the rest of the non-linear dynamics of the Heisenberg model. It is instructive to note that the full dynamics of the isotropic Heisenberg ferromagnet ($\Delta=1$) has also been studied by means of a NLS equation in the classical [18-20] and quantum [21-23] cases. This approach is manifestly different from ours, because the continuum limit is done in a different fashion (within our procedure the isotropic Heisenberg ferromagnet is mapped onto a free boson field theory and the coupling constant of our NLS model is proportional to the parameter $\Delta-1$, measuring the deviation from isotropy!) The various possible approaches show that the NLS model has several non-isomorphous applications in the study of Heisenberg spin chains.

The anisotropic exchange operators for SU($\mathcal{N}$), the continuum limit of which we have shown to be the ($\mathcal{N}-1$)-component NLS models, are novel. Representing the SU($\mathcal{N}$) generators as products of SU(2) generators, spin chain models with tensor interactions arise. Isotropic exchange of this type ($\lambda = 1$) has been studied by several authors [24-28] in various contexts. These isotropic models—sometimes called Schrödinger Hamiltonians—are exactly soluble by means of a nested Bethe ansatz. For our anisotropic models it can be shown that the nested Bethe ansatz works for $\lambda = \pm 1, 0$. For $\lambda \neq \pm 1, 0$ it works only for the completely symmetric and completely antisymmetric representations of the permutation group S_N. For other representations of S_N the triangle equation cannot be verified using exponential functions. Transfer matrix techniques must be applied in order to obtain the general solution for arbitrary λ. Details will be published elsewhere.

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